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Bose condensation at high temperatures

Hidenori SONODA¹

Physics Department, Kobe University, Kobe 657-8501, Japan

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Abstract

Bose condensation is usually a low temperature phenomenon due to a low particle number density. When the number density is kept large compared to the inverse Compton volume, Bose condensation can occur at a temperature much higher than the mass of the particle. We can then use a three dimensional effective theory to study the thermal properties. We compute the transition temperature for a complex scalar field theory with a small interaction parameter.

In a previous letter [1] we used an effective three dimensional field theory [2, 3] to compute the critical temperature of the weak coupling ϕ^4 theory. The purpose of this letter is to extend the same technique to study theories with a chemical potential coupled to a conserved particle number. We will compute the transition temperature T_c for Bose condensation and the temperature dependence of the specific heat near T_c .

It is the low number density of particles which makes Bose condensation a low temperature phenomenon. If the number density is large compared to the inverse Compton volume m^3 , where m is the particle mass, the transition temperature T_c for Bose condensation becomes much higher than m so that T_c is calculable using a three dimensional effective theory.

In order to see this possibility more quantitatively, we consider a free theory of spinless particles and antiparticles of mass m . The particle number density n (or more precisely the number density of particles minus that of antiparticles) is an increasing function of the chemical potential μ . At the maximum chemical potential $\mu = m$, we obtain the maximum number density without Bose condensation. For temperature T much higher than

¹E-mail: sonoda@phys.sci.kobe-u.ac.jp

m , the maximum number density is approximately

$$n \simeq \frac{mT^2}{2} + \frac{4m^3}{3(4\pi)^2} . \quad (1)$$

For n larger than the right-hand side, Bose condensation takes place. Hence, for a given n , the transition temperature T_c is given by

$$T_c^2 \simeq \frac{2}{m} \left(n - \frac{4m^3}{3(4\pi)^2} \right) . \quad (2)$$

This implies $T_c \gg m$ for $n \gg m^3$. The above result (2) is expected to be valid with a small correction in the presence of weak interactions among the bosons.

For concreteness we consider a complex scalar field theory with a global U(1) symmetry in thermal equilibrium at temperature T . The lagrangian density in the four dimensional euclidean space is given by

$$\mathcal{L}_0 = \partial_\mu \phi^* \partial_\mu \phi + m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 + (\text{counterterms}) , \quad (3)$$

where ϕ is a complex scalar field which is periodic in the euclidean time direction:

$$\phi(\vec{x}, \tau + 1/T) = \phi(\vec{x}, \tau) . \quad (4)$$

The counterterms are given in the $\overline{\text{MS}}$ scheme. Hence, the renormalized parameters satisfy the renormalization group (RG) equations

$$\frac{d}{dt} \frac{\lambda}{(4\pi)^2} = -5 \left(\frac{\lambda}{(4\pi)^2} \right)^2 + 15 \left(\frac{\lambda}{(4\pi)^2} \right)^3 + \dots , \quad (5)$$

$$\frac{d}{dt} m^2 = \left(2 - 2 \frac{\lambda}{(4\pi)^2} + \frac{5}{2} \left(\frac{\lambda}{(4\pi)^2} \right)^2 + \dots \right) m^2 . \quad (6)$$

We can introduce the chemical potential μ as the euclidean time component of an external U(1) gauge field:

$$A_\tau = i\mu . \quad (7)$$

Note this is purely imaginary. The Ward identity protects it from renormalization. The total lagrangian density is therefore

$$\mathcal{L} = \mathcal{L}_0 - \mu Z \phi^* \overleftrightarrow{\partial}_\tau \phi - \mu^2 Z \phi^* \phi , \quad (8)$$

where Z is the wave function renormalization constant. The average number density of particles, n , is the first order derivative of the free energy density $Y_4(T, \mu)$ with respect to μ :

$$n = - \left(\frac{\partial Y_4}{\partial \mu} \right)_T . \quad (9)$$

Our task is to find the transition temperature T_c so that for $T < T_c$ the field ϕ gets a non-vanishing expectation value. We will find T_c first for a given chemical potential μ , and then for a given number density n .

A naive loop expansion of Y_4 suffers from infrared divergences at two-loop and beyond [4]. This difficulty is best avoided by reducing the theory to a three dimensional effective theory [2, 3] whose infrared properties are much better understood [5, 6]. The effective theory is given by the following lagrangian density:

$$\mathcal{L}_3 = g_3 + \partial_\mu \phi_3^* \partial_\mu \phi_3 + m_3^2 \phi_3^* \phi_3 + \frac{\lambda_3}{4} (\phi_3^* \phi_3)^2 + (\text{counterterms}) , \quad (10)$$

where the counterterms are given in the $\overline{\text{MS}}$ scheme. The higher dimensional interaction terms are negligible within our approximation. The RG equations of the parameters are given by

$$\frac{d}{dt} g_3 = 3g_3 + \tilde{A} \lambda_3^3 , \quad \frac{d}{dt} m_3^2 = 2m_3^2 + \tilde{C} \lambda_3^2 , \quad \frac{d}{dt} \lambda_3 = \lambda_3 , \quad (11)$$

where

$$\tilde{A} = \frac{1}{(4\pi)^2} \frac{5}{3 \cdot 2^9} , \quad \tilde{C} = -\frac{1}{(4\pi)^2} \frac{1}{2} . \quad (12)$$

The parameters of the two theories are related such that the free energy density $F_3(g_3, m_3^2, \lambda_3)$ of the effective theory reproduces that of the original theory:

$$Y_4(T, \mu) = T F_3(g_3, m_3^2, \lambda_3) . \quad (13)$$

For the three dimensional reduction to be valid, the temperature must be high compared to m , μ :

$$T \gg m \approx \mu . \quad (14)$$

Then, we can choose a renormalization scale Λ such that

$$T/\Lambda = \mathcal{O}(N) , \quad m^2/\Lambda^2 = \mathcal{O}(N) , \quad \mu^2/\Lambda^2 = \mathcal{O}(N) , \quad (15)$$

where $O(N)$ denotes an order of N , a large number. We consider such a range of T so that we can identify the smallness of $1/N$ with the smallness of the coupling λ :

$$\lambda = O(1/N) . \quad (16)$$

Eqns. (15,16) guarantee that we can calculate the parameters of the effective theory in powers of λ , m^2/T^2 , and μ^2/T^2 which are all of order $1/N$. Since the calculation is straightforward, we omit the detail and only state the result:

$$\lambda_3 \simeq \lambda T , \quad (17)$$

$$\begin{aligned} m_3^2 \simeq & m^2 - \mu^2 + \frac{\lambda}{12} T^2 + \frac{\lambda}{(4\pi)^2} \left[2m^2(\ln T/\Lambda + j_2) - 2\mu^2 \right] \\ & + \frac{\lambda^2}{(4\pi)^2} T^2 \left[-\frac{1}{12} \ln T/\Lambda + \frac{j_2}{6} - \frac{j_3}{4} \right] , \end{aligned} \quad (18)$$

$$\begin{aligned} Tg_3 \simeq & T^4 \frac{(4\pi)^2}{144} \left[-\frac{1}{5} + \frac{1}{2} \frac{\lambda}{(4\pi)^2} + \left(\frac{\lambda}{(4\pi)^2} \right)^2 \left(\frac{5}{2} \ln T/\Lambda + j_2 + \frac{3}{2} j_4 \right) \right] \\ & + \frac{T^2}{12} \left[m^2 \left(1 + 2 \frac{\lambda}{(4\pi)^2} (\ln T/\Lambda + j_2) \right) + \mu^2 \left(-3 - 2 \frac{\lambda}{(4\pi)^2} \right) \right] \\ & + \frac{1}{(4\pi)^2} \left[m^4 (\ln T/\Lambda + j_2) + \frac{2}{3} \mu^4 - 2m^2 \mu^2 \right] , \end{aligned} \quad (19)$$

where the constants are given by²

$$\begin{aligned} j_2 &= \ln 4\pi - \gamma , \quad j_3 = \ln 4\pi - 1 - \frac{\zeta'(-1)}{\zeta(-1)} , \\ j_4 &= \ln 4\pi - \frac{31}{30} - 2 \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-3)}{\zeta(-3)} . \end{aligned} \quad (20)$$

In the above we have computed $\lambda_3, m_3^2/\Lambda^2$ to order N^0 and Tg_3/Λ^4 to order N^2 .

Let us determine the transition temperature T_c as a function of the chemical potential μ . In the effective theory the expectation value $\langle \phi_3 \rangle \propto \langle \phi \rangle$ is non-vanishing if

$$R(m_3^2, \lambda_3) < R_c , \quad (21)$$

where $R(m_3^2, \lambda_3)$ is an RG invariant defined by [1, 8]

$$R(m_3^2, \lambda_3) \equiv \frac{m_3^2}{\lambda_3^2} - \tilde{C} \ln \lambda_3 . \quad (22)$$

²The necessary integrals have been calculated by Arnold and Zhai [7].

Only a non-perturbative calculation can determine R_c , and we must leave it as an unknown constant here. By substituting eqns. (17,18) into $R = R_c$, we obtain the transition temperature [1]:

$$\begin{aligned} \frac{\lambda}{12} T_c^2 \simeq \mu^2 - m^2 \\ + \frac{\lambda}{(4\pi)^2} \left[12(\mu^2 - m^2) \left((4\pi)^2 R_c - \frac{1}{2} \ln \lambda - \frac{5}{24} \ln \frac{12(\mu^2 - m^2)}{\lambda \Lambda^2} + \frac{j_3}{4} - \frac{j_2}{6} \right) \right. \\ \left. - m^2 \ln \frac{12(\mu^2 - m^2)}{\lambda \Lambda^2} - 2m^2 j_2 + 2\mu^2 \right], \end{aligned} \quad (23)$$

where the constants j_2, j_3 are given by eqns. (20). The above gives T_c/Λ to order N^0 . The dependence on $\ln \lambda$ is the source of infrared divergences in the naive loop expansions.

Eqn. (23) gives T_c as a function of the chemical potential μ , but it is more convenient to express T_c as a function of the number density n . To do this, we must express μ as a function of n by inverting eqn. (9). The equivalence (13) implies that we must obtain the free energy density F_3 of the effective theory. Since the cosmological constant g_3 has nothing to do with interactions, we find

$$F_3(g_3, m_3^2, \lambda_3) = g_3 + f_3(m_3^2, \lambda_3). \quad (24)$$

The RG eqns. (11) imply that the function f_3 can be written as

$$f_3(m_3^2, \lambda_3) = \lambda_3^3 \left(-\tilde{A} \ln \lambda_3 + \overline{f}_3(R) \right), \quad (25)$$

where \tilde{A} is given in eqns. (12), and \overline{f}_3 is a function of the RG invariant R (22). The theory of critical phenomena gives the following scaling formula near $R = R_c$ [8, 9]:

$$\overline{f}_3(R) = \overline{f}_3(0) + a |R - R_c|^{\frac{3}{y_E}} + O\left(|R - R_c|^{\frac{3-y'}{y_E}}\right), \quad (26)$$

where $a > 0$ is a constant. The constants y_E and y' are the critical exponents of the three dimensional XY model: $y_E > 0$ is the scale dimension of the relevant parameter, and $y' < 0$ is that of the least irrelevant parameter. They are given approximately as [9]

$$y_E \simeq 1.6, \quad y' \simeq -0.4. \quad (27)$$

Substituting the above results into eqn. (9), we obtain the relation between n and μ

$$n = 2\mu T^2 \left[\frac{1}{4} - \frac{1}{(4\pi)^2} \frac{\frac{4}{3}\mu^2 - 2m^2}{T^2} + \frac{\lambda}{(4\pi)^2} \left(\frac{1}{6} + (4\pi)^2 \overline{f}_3'(R) \right) \right] \quad (28)$$

up to terms of order $N^{\frac{3}{2}}\Lambda^3$. Note that the derivative $\overline{f}_3'(R)$ vanishes at criticality $R = 0$.

Solving eqns. (23,28), we can obtain the critical temperature T_c as a function of the number density n to the order $N^0\Lambda$. For simplicity, however, we will only present the result at the leading order which is $N\Lambda$. Eqns. (23,28) give

$$\frac{1}{12}\lambda T_c^2 = \mu^2 - m^2, \quad n = \frac{1}{2}\mu T_c^2. \quad (29)$$

These give a cubic equation for $X \equiv \frac{mT_c^2}{2n}$:

$$\nu X^3 = 1 - X^2, \quad (30)$$

where $\nu \equiv \frac{\lambda n}{6m^3}$ is a dimensionless constant of order N^0 . Let $X_0(\nu)$ be the solution that lies between 0 and 1. (See Fig. 1.) Then, we obtain

$$T_c^2 = \frac{12m^2}{\lambda} \nu X_0(\nu), \quad \mu^2 - m^2 = m^2 \nu X_0(\nu). \quad (31)$$

The function $\nu X_0(\nu)$ increases as ν , and it behaves as $\nu^{\frac{2}{3}}$ for $\nu \gg 1$ and $\nu - \frac{\nu^2}{2}$ for $\nu \ll 1$. For a very small coupling such that $\nu \ll 1$, we find

$$T_c^2 \simeq \frac{2n}{m} - \lambda \frac{n^2}{6m^4}. \quad (32)$$

Therefore, the critical temperature decreases as the coupling increases. This tendency also exists in the low temperature Bose condensation phenomena.

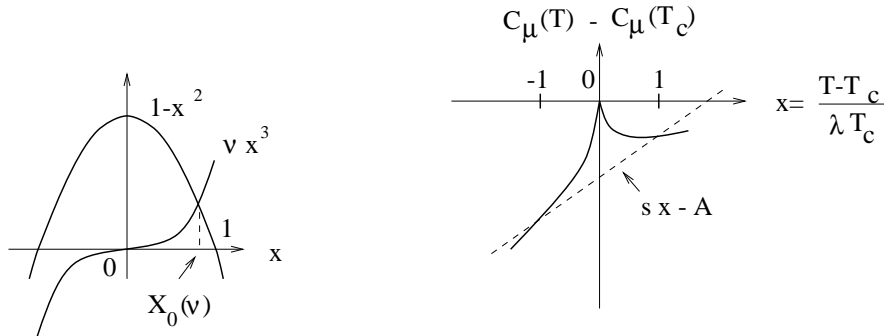


Fig. 1 Solution of the cubic equation (30)

Fig. 2 Specific heat near the critical temperature

Finally, let us compute the temperature dependence of the specific heat near the critical temperature T_c . Using the free energy density of the three dimensional effective theory given by eqns. (24–26), we obtain

$$\begin{aligned} C_\mu(T, \mu) &\equiv -T \frac{\partial^2 Y_4(T, \mu)}{(\partial T)^2} = -T \frac{\partial^2 (TF_3)}{(\partial T)^2} \\ &\simeq C_\mu(T_c, \mu) + s \frac{T - T_c}{\lambda T_c} - A \left| \frac{T - T_c}{\lambda T_c} \right|^{-\alpha}, \end{aligned} \quad (33)$$

where

$$C_\mu(T_c, \mu) = \frac{(4\pi)^2}{60} T_c^3 + \frac{1}{3} m^2 T_c, \quad (34)$$

$$s = \frac{(4\pi)^2}{20} \lambda T_c^3, \quad A = \frac{1}{6^{\frac{3}{y_E}} y_E} \left(\frac{3}{y_E} - 1 \right) a \lambda T_c^3, \quad (35)$$

and

$$\alpha \equiv 2 - \frac{3}{y_E} \simeq -0.01. \quad (36)$$

The above expression (33) is valid for $|T - T_c|/(\lambda T_c) = O(N^0)$, and we have ignored the terms of order $N\Lambda^3$. By keeping only the first two terms of eqn. (26), we have also ignored $(|T - T_c|/(\lambda T_c))^{-y'/y_E} \ll 1$. The positive constant a in A is the unknown constant in eqn. (26). In Fig. 2 we show the specific heat schematically. The specific heat at constant number density, C_n , differs from C_μ only by a constant up to order $N^2\Lambda^3$:

$$C_n - C_\mu \simeq -2\mu^2 T_c \simeq -2T_c \left(m^2 + \frac{\lambda}{12} T_c^2 \right). \quad (37)$$

In conclusion we have shown how to apply the method of three dimensional effective theory to understand Bose condensation at high temperatures.

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